

## Missing equation in Piketty's $r$ - $g$ theory

Yasunori Fujita\*

*Keio University, Japan*

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### Abstract

Although Thomas Piketty has revolutionized our understanding of inequality, logic behind his  $r$ - $g$  theory is not developed formally in his researches. The present paper, by finding out the missing part of Piketty's  $r$ - $g$  theory, attempts to construct a model to show the condition where  $\beta$  (capital/income ratio) increases over time when  $r$  (rate of return on capital) is greater than  $g$  (growth rate of output). It is revealed that necessary and sufficient condition for the  $r$ - $g$  theory is  $S_Y(t) = C_F(t)$ . That is, savings (left hand side) are equal to the consumption using financial capital (right hand side). We also reveal that the problem is not in the Second Fundamental Law of Capitalism, but in the formulation of capital accumulation dynamics. Furthermore, it is demonstrated that the two Fundamental Laws of Capitalism are not necessary for deriving the  $r$ - $g$  theory.

*Keywords:* Second Fundamental Law of Capitalism, capital accumulation, inequality

*JEL Classification Codes:* D31, E10, O20, P10

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### 1. Introduction

Our understanding of inequality has been changed drastically by Thomas Piketty's extensive studies that connect wealth concentration to the  $r > g$  inequality, where  $r$  and  $g$  denote rate of return on capital and growth rate of output, respectively (Alvaredo, Atkinson, Piketty, and Saez (2013), Piketty (2011, 2014, 2015), Piketty and Saez (2003, 2014), Piketty and Zucman (2014) etc.). Since Piketty and his associates, overlooking Marx (1867) and Kuznets (1955), shed new light on the huge amount of studies of economic growth, their influence has been running deep as Paul Krugman mentions (2014*a,b*). In fact, according to Jones (2015), Piketty is so popular that the algebra  $r > g$  is seen even on T-shirts.

Despite this "Piketty Panic", underlying logic behind the  $r$ - $g$  theory is not developed formally in his researches. It is true that Piketty shows that coefficient of Pareto distributions measuring inequality is an increasing function of the gap  $r$ - $g$  (Atkinson, Piketty and Saez (2011)) and, of

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\* E-mail: yfujita@econ.keio.ac.jp.

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course, Pareto distribution is widely utilized to analyze the inequality (Atkinson and Harrison (1978), Stiglitz (1969) and Jones (2015)). It would be, however, better to prove the  $r$ - $g$  theory in relation to the capital/income ratio  $\beta$ , the most important variable in Piketty's framework that expresses the income-wealth-inequality. In other words, since Piketty's  $r$ - $g$  theory demonstrates that capitalism generates inequalities (*i.e.* increases  $\beta$ ) when  $r$  exceeds  $g$ , what is necessary is to construct a model where  $\beta$  increases over time when  $r$  is greater than  $g$ . In Piketty's recent seminal work (Piketty (2014)) also, although importance of the capital/income ratio  $\beta$  and the gap  $r$ - $g$  is shown respectively in FIGURE I.2. (The capital/income ratio in Europe, 1870–2010) and FIGURE 10.9. (Rate of return versus growth rate at the world level, from Antiquity until 2100), these two figures are not explained within the context of their relationship.

As is shown in section 3 (Capital Accumulation Dynamics and the  $r$ - $g$  Theory), if we postulate the Second Fundamental Law of Capitalism and capital accumulation dynamics as in most of the works of Piketty and his colleagues (Piketty (2014) etc.), derivative of  $\beta$  with respect to time turns out to be zero (*i.e.* the capital/income ratio remains constant over time) for every  $r$  and  $g$ , which, however, is *not* what the  $r$ - $g$  theory should be. Thus, we have to admit something is missing in Piketty's way of analyzing.

In what follows, by finding out the missing part of the  $r$ - $g$  theory and dividing the capital into two categories (*i.e.* real capital and financial capital), we try to construct a model, to show the condition where  $\beta$  increases over time when  $r$  is greater than  $g$ . As we will see, main issue stems from the differentiation between real and financial capital. It is revealed that necessary and sufficient condition for the  $r$ - $g$  theory is  $S_Y(t) = C_F(t)$ . That is, savings (left hand side) are equal to the consumption using financial capital (right hand side). It is also revealed that the problem is not in the Second Fundamental Law of Capitalism, which has been subjected to severe criticism (Krusell and Smith (2014)), but in the formulation of capital accumulation dynamics. Furthermore, it is demonstrated that the two Fundamental Laws of Capitalism are *not* necessary for deriving the  $r$ - $g$  theory.

There have been some papers dealing extensively with the  $r > g$  inequality. Bernardo, Martínez and Stockhammer (2014) examine if  $r > g$  is necessarily associated with increasing inequality, while Rowthorn (2014) investigate if the rising income share of wealth-owners is due to the over-accumulation of capital. Dumenil and Levy (2014), paying attention to the definition of capital and focusing on fixed capital, analyze if the existence of a tendency toward the gradual concentration of wealth in capitalism is likely or not.

By constructing a simple theoretical model, we attempt to shed new light on this field.

## 2. Basic model

A controversial aspect of *Capital in the 21st Century* is Piketty's definition of capital itself. So, let us start with the definition of capital.

According to Piketty, capital is defined as total sum of nonhuman assets that can be owned and exchanged on markets, which include all forms of real capital *and* financial capital. In Piketty's formulation, however, distinction of these two types of capital is ambiguous. As is often the case with Piketty's studies (Alvaredo, Atkinson, Piketty, and Saez (2013), Piketty (2011, 2015) etc.), *Capital in the 21st Century* defines the capital/income ratio  $\beta$  as  $K/Y$  and, *at the same time*, formulates process of production as a function of capital  $K$  and labor  $L$  (*e.g.* Cobb-Douglas production function,  $Y = K^\alpha L^{1-\alpha}$ , where  $\alpha$  is a positive parameter). This formulation, coupled with

that fact that amount of production is equal to national income in standard macroeconomics (Keynes (1936), Samuelson (1948), etc.) implies immediately that all the capital is used *directly* for production, which is not proper representation of Piketty's definition that includes financial capital, since financial capital is *not* directly employed in the production process.

In the following, in order to formulate Piketty's definition, we explicitly divide the capital into two categories:

- (1) real capital (capital that is employed in the production process);
- (2) financial capital (capital that is *not* employed in the production process).

Let us assume that real capital of size  $K(t-1)$  in period  $t-1$  produces goods and services in period  $t$ ,  $Y(t)$ , which is equal to national income in period  $t$ . We also assume that in period  $t$ , the national income of size  $Y(t)$  is used either for consumption,  $C_Y(t)$  or savings for accumulation of real capital,  $S_Y(t)$ .

The above allocation of national income is expressed as

$$Y(t) = C_Y(t) + S_Y(t) \quad (1)$$

Let us also assume financial capital in period  $t-1$ ,  $F(t-1)$ , becomes  $(1+r)F(t-1)$  in period  $t$ , which is used in period  $t$  either for consumption,  $C_F(t)$ , savings for accumulation of real capital,  $S_F(t)$ , or demand for financial capital,  $F(t)$ . Throughout this paper, for the simplicity of analysis, we treat  $r$  as an exogenous parameter.

We can express the above allocation of capital income as

$$(1+r)F(t-1) = C_F(t) + S_F(t) + F(t) \quad (2)$$

### 3. Capital accumulation dynamics and the *r-g* theory

We assume that real capital in period  $t-1$ ,  $K(t-1)$ , plus savings becomes the real capital in the next period and  $K(t-1)$  is held through corporate shares, which is counted as financial capital. If we let  $F_k(t-1)$  denote the financial capital through which  $K(t-1)$  is held, it follows that  $K(t-1) = F_k(t-1)$ . Therefore, the total sum of capital in period  $t$  becomes  $K(t) + F(t) - F_k(t-1)$  since  $K(t-1)$  and  $F_k(t-1)$  are double counted.

If we assume real capital does not depreciate as in most of Piketty's studies (Piketty (2014) etc.) accumulation dynamics of real capital is expressed as

$$K(t) = K(t-1) + S_Y(t) + S_F(t) \quad (3)$$

As for accumulation dynamics of financial capital, on the other hand, we have

$$F(t) = (1+r)F(t-1) - C_F(t) - S_F(t) \quad (4)$$

directly from (2). Therefore, if we let  $W(t)$  denote the total sum of capital in period  $t$ , we obtain its accumulation dynamics as

$$W(t) = (1+r)W(t-1) + S_Y(t) - C_F(t), \quad (5)$$

which reduces to the following derivative of  $W$  with respect to time,  $t$

$$\frac{dW(t)}{dt} = rW(t) + S_Y(t) - C_F(t), \quad (6)$$

by assuming continuous time horizon for the simplicity of analysis.

Substituting (6) and the definition of  $g$  ( $\equiv \frac{dY(t)}{Y(t)}$ ) into the derivative of  $\beta$  ( $\equiv \frac{W(t)}{Y(t)}$ ) (i.e.

$$\frac{d\beta}{dt} = \frac{\frac{dW(t)}{dt} Y(t) - W(t) \frac{dY(t)}{dt}}{\{Y(t)\}^2}, \text{ we have the dynamics of the capital/income ratio } \beta \text{ as}$$

$$\frac{d\beta}{dt} = \frac{1}{Y(t)} \{(r-g)W(t) + S_Y(t) - C_F(t)\}. \quad (7)$$

This equation, the missing part of the  $r$ - $g$  theory, enables us to derive the condition where  $\beta$  increases over time when  $r$  is greater than  $g$ .

*Proposition:*

*Necessary and sufficient condition for the  $r$ - $g$  theory is  $S_Y(t) = C_F(t)$ .*

Here, the  $r$ - $g$  theory is defined as a theory that states  $\beta$  increases over time when  $r$  is greater than  $g$ .

Since  $S_Y(t)$  is savings, we can say that  $S_Y(t) = C_F(t)$  describes a situation where savings (left hand side) are equal to the consumption using financial capital (right hand side).

More understandable situation for  $S_Y(t) = C_F(t)$  is  $S_Y(t) = C_F(t) = 0$ . Since  $S_Y(t) = Y(t) - C_Y(t)$  holds by the above allocation of national income,  $S_Y(t) = 0$  is equivalent to  $Y(t) = C_Y(t)$ , which implies that  $S_Y(t) = C_F(t) = 0$  is a situation where all the national income is used for consumption, while no income from the financial capital is used for consumption.

Note that if we assume, as in Piketty (2014) etc.,  $W = K$  (i.e. financial capital is assumed away),

$$\beta \equiv \frac{K}{Y} = \frac{s}{g}, \text{ where } s \text{ stand for savings rate (i.e. the Second Fundamental Law of Capitalism) and}$$

$$\frac{dK}{dt} = sY \quad (\text{i.e. capital accumulation dynamics}), \text{ it follows that}$$

$$\frac{d\beta}{dt} = \frac{dK(t)}{dt} \frac{1}{Y(t)} - \frac{K(t)}{Y(t)} \frac{dY(t)}{dt} = sY(t) \frac{1}{Y(t)} - \frac{s}{g} g = 0.$$

It also follows from (7) that if  $S_Y(t) > C_F(t)$ ,  $r > g$  is a sufficient condition (not a necessary condition) for  $\frac{d\beta}{dt} > 0$ . Since  $S_Y(t) = Y(t) - C_Y(t)$  as we mentioned above,  $S_Y(t) > C_F(t)$  is equivalent

to  $Y(t) > C_Y(t) + C_F(t)$ , which means that the output is greater than the total consumption. Since this condition always holds if we presume a closed economy or the world economy as a whole, we have the following corollary.

*Corollary:*

*In a closed economy or in the world as a whole,  $\frac{d\beta}{dt} > 0$  holds if  $r > g$ .*

López-Bernardo et al. (2014) showed that a constant capital-output ratio and even a decreasing one is perfectly compatible with  $r > g$  within a framework of Cambridge model. From (7), we can say that  $\frac{d\beta}{dt} = 0$  and  $r > g$  if  $(r - g)W(t) = C_F(t) - S_Y(t)$  and  $\frac{d\beta}{dt} < 0$  and  $r > g$  if  $(r - g)W(t) < C_F(t) - S_Y(t)$ , which is consistent with López Bernardo et al. (2014). This corollary, however, shows that in a closed economy or in the world as a whole, unlike López Bernardo et al. (2014), a constant capital-output ratio or a decreasing one is *not* compatible with  $r > g$ .

#### 4. Conclusion

We began this research with the hope of showing the condition where  $\beta$  increases over time when  $r$  is greater than  $g$ . Our results were more surprising than we had hoped: the two Fundamental Laws of Capitalism, which are two of the most essential formulas in Piketty's economics, are *not* necessary for deriving the  $r$ - $g$  theory, and the  $r$ - $g$  theory remains intact as a sufficient condition.

A great deal of controversy has been provoked by Piketty's economics (Milanovic, 2014). We truly hope this research note, which finds out the missing part of the  $r$ - $g$  theory, will contribute to better understanding of the new and rich framework Thomas Piketty has provided.

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