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# A micro-foundation of a simple financial model with finite-time singularity bubble and its agent-based simulation

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**Abstract**

This paper proposes a mathematical model of financial security prices in continuous time with bubbles in which prices may diverge and crash in finite time. Just before the bubbles burst, prices increase super-exponentially. In addition, a discrete-time excess demand model is proposed to provide a micro-foundation for the continuous-time model. The derived discrete-time security price model has the same characteristics as the continuous-time price model and expresses the finite-time singularity. Furthermore, based on the excess demand model, an agent-based simulation is performed to check the price behavior. As expected, we can confirm that prices can diverge in finite time and increase super-exponentially.

*Keywords*: Financial market; Bubble; Stochastic model; Agent-based simulation
*JEL Classification Codes*: G12, G17

## 1. Introduction

The Johansen-Ledoit-Sornette (JLS) model of rational expectation bubbles with the finite-time singular crash and log-periodic oscillations has been developed to describe the dynamics of financial bubbles and crashes. Note that the finite-time singularity of the price means that the price diverges to infinity in finite time. Detailed descriptions and references can be found in survey papers, e.g., Sornette et al. (2013).

In that model, the deterministic approximate curve of the prices at time $t$ is presented by:

$$p(t) \approx A + B(t_c - t)^\beta + C(t_c - c)^\beta \cos(\omega \ln(t_c - c) + \phi)$$

where $A, B, C, \beta, \omega, \phi$ are constant parameters and $t_c$ denotes the time when the price will crash. When $\beta < 0$, the second term in the right-hand side of the above equation expresses the *finite-time singularity* of the price. From this, the bubble in this model can be regarded as a super-exponential bubble, i.e., the bubble where the price grows faster than any exponential function.

In Lin and Sornette (2013), a stochastic differential equation (SDE) model of security prices with finite-time singularity bubbles is proposed. They have modeled the crash time predicted by the agent as a stochastic process which reverts to the mean of the actual crash time, and by

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adding a positive feedback effect to the price, they created a self-reinforcing price model and succeeded in expressing the finite-time singularity. The fundamental idea used in the modeling is the analogy of that, if $m > 1$, solutions of the ordinary differential equation:

$$\frac{dp(t)}{p(t)} = cp(t)^{m-1}dt$$

have the finite-time singularity unlike in the case where $m = 1$. The quantity $p(t)^{m-1}$ on the right-hand side indicates that the price has a positive feedback effect on the return $dp/p$. Note that when $m = 1$, the return has no positive feedback effect from the price and becomes an exponential dynamics price model like the Black-Scholes model.

The purpose of this paper is also to propose a bubble model with the finite-time singularity. The difference between our model and Lin and Sornette (2013) is the following two points. First, ours does not assume the positive feedback effect from prices on returns. This makes the representation of the logarithmic return of the price simpler. Second, our agents predict not the time when crash occur but the remaining time until the crash. The model of the predicted remaining time to crash is not a mean-reverting stochastic process but a diffusion process with a constant drift coefficient and a diffusion coefficient proportional to the process itself. The shorter the agents' predicted remaining time to crash, the closer the diffusion coefficient approaches zero.

Next, using the continuous-time price model made as a reference, a microeconomic market model in discrete time will be considered to propose a micro-foundation of the finite-time singularity. For the method of modeling the micro-foundation, the excess demand model of Föllmer and Schweizer (1993) is employed here. The excess demand model of Föllmer and Schweizer (1993) assumes that agents have their own excess demand functions of the security for the proposed price, and the price is determined to satisfy the market clearing condition that the sum of the excess demands over all agents must be zero. This study has been extended in several ways. In the model of Rheinlaender and Steinkamp (2004), excess demands for fundamentalists and momentum traders in the market are considered, and the phenomenon of endogenous phase transitions in market stability is studied. In Farkas et al. (2017), a micro-foundation of a stochastic volatility model is proposed by setting up an excess demand model of rational agents and irrational agents who overreact to price changes assuming a herding effect on the ratio of the number of these agents. Like these, the current paper has also employed an excess demand model for the micro-foundation. In our setting, the excess demands of agents approach zero when the mean of the agents' predicted remaining time to crash is shortened.

Next, based on the created excess demand model, an agent-based simulation is performed to examine the behavior of the excess demand model. It may be able to say that the ability to perform agent-based simulations is one of the advantages of considering micro-foundations in pricing models. There already have existed agent-based models of asset prices that include bubbles such as Cividino et al. (2023). It has explained how the interaction between rational fundamentalists and trend-following noise traders leads to bubbles and crashes through the agent-based simulation. In the current paper, our agents incorporate the predicted remaining time to crash into their investment decisions and it can be observed that this sometimes leads to the occurrence of finite-time singularity.

The rest of the paper is organized as follows. Agents' prediction of the remaining time to crash of the price and the continuous-time security price model derived from them is proposed in section 2. Section 3 provides a micro-foundation of the model made in the previous section as a microeconomic excess demand model. In section 4, the agent-based simulation describing the excess demand model is performed. Section 5 concludes the paper.

## 2. Continuous time-model

Suppose that one financial security is traded in the market. The price of the security may show an irrational increase, i.e., a bubble.

Each agent participating in the trade predicts the remaining time until the price diverges to infinity or the bubble crashes. $\hat{X}_t$ average of the predicted periods over all agents and it is assumed that $\hat{X}_t$ satisfies:

$$\mathrm{d}\hat{X}_t = f\hat{X}_t \mathrm{d}W_t - c\mathrm{d}t, \quad t < \tau_c \; (1)$$

with $\hat{X}_0 > 0$ where $\tau_c := \inf\{t > 0 | \hat{X}_t = 0\}$ and $c, f > 0$. That is,

$$\hat{X}_t = e^{fW_t - \frac{f^2 t}{2}}\left(\hat{X}_0 - c\int_0^t e^{\frac{f^2 u}{2} - fW_u}\,\mathrm{d}u\right)$$

(see Eq. (4.9) in chapter 4 of Kloeden and Platen, 1992). It is expressed that the shorter the average of agents' predicted time until the crash becomes, the closer the diffusion coefficient approaches zero, itself decreasing with a higher probability.

The security price $S_t$ at time $t$ is supposed to be determined from this as:

$$\hat{S}_t = a\hat{X}_t^{-b}, \qquad t < \tau_c$$

where $a, b > 0$. Since $\hat{S}_t \to \infty$ when $t \uparrow \tau_c$, it can be seen that the price may diverge to infinity in finite time. For later use, the SDE satisfied by the logarithmic price of this security price is derived here. Applying the Ito's formula to $\ln \hat{S}_t = \ln a - b \ln \hat{X}_t$, we get:

$$\mathrm{d}\ln \hat{S}_t = -b\left(\hat{X}_t^{-1}\mathrm{d}X_t - \frac{1}{2}\hat{X}_t^{-2}(\mathrm{d}X_t)^2\right)$$

Then, substituting Eq. (1) into this, we obtain:

$$\mathrm{d}\ln \hat{S}_t = -b\left(f\mathrm{d}W_t - \frac{c}{\hat{X}_t}\mathrm{d}t - \frac{1}{2}f^2\mathrm{d}t\right) (2)$$

## 3. Micro-foundation of the finite-time singularity

Referring to the price model provided in the previous section, a micro-foundation of the finite-time singularity is given by an excess demand model in the same spirit as Föllmer and Schweizer (1993).

Suppose that there are $N \in \mathbb{N}$ agents in the trading of the security and they each have their own excess demand function. The excess demand functions represent the desired quantity of transactions that agents would demand or supply if the price at the time was proposed to be some given value. The set of all agents is denoted by $A$ and the excess demand of an agent $\alpha \in A$ at time $k = 0,1,2,\cdots$ for the proposed price $s > 0$ is written by $e_{\alpha,k}(s)$. The price of the security is assumed to be determined such that the sum of agents' excess demand becomes zero. This condition is called the *market-clearing* condition. The market-clearing condition means that the price would be determined in such a way that the desired quantity is balanced over all agents. Therefore, the security price at time $k$ is determined as the solution of the equation:

$$\sum_{\alpha \in A} e_{\alpha,k}(S_k) = 0.$$

Please note that the time stamp here is discrete unlike in the previous section.

Next, the excess demands of agents are specified. Each agent predicts the remaining time until the price crashes and the period predicted by $\alpha \in A$ just after the price $S_k$ at time $k = 0,1,2,\cdots$ is determined is assumed to be:

$$X_{\alpha,k+1} = X_{\alpha,k} - q + p\bar{X}_k \epsilon_{\alpha,k+1}$$

with $X_{\alpha,0} > 0$ where $p, q > 0$ are constants, $\bar{X}_k$ is a modification of the average of the predictions defined later and $\epsilon_{\alpha,k}, \alpha \in A, k = 1,2,\cdots$ are independent and identically distributed random variables with mean zero and variance one. That is, each agent predicts the remaining time to crash to essentially become shorter constantly, but with some degree of randomness, depending on the market's average predicted time. Note that this value is not necessarily an integer. Negative values of the predictions are regarded as those agents alerting the market by lowering the average in the market. The average of these predictions is written by:

$$\tilde{X}_k = \frac{1}{N}\sum_{\alpha\in A} X_{\alpha,k}\,, \qquad k = 0,1,2,\cdots$$

and define: $\bar{X}_k = \max\{\tilde{X}_k, 0\},\ \ k = 0,1,2,\cdots$. Also define $T_c \coloneqq \inf\{k | \bar{X}_k = 0\}$. Then, the agents' excess demand functions are set by:

$$e_{\alpha,k+1}(s) = -X_{\alpha,k}\ln\frac{s}{S_k} + r\{(1+v)X_{\alpha,k} - X_{\alpha,k+1}\}$$

where $v, r > 0$. That is, agents first observe the logarithmic returns regarding the proposed price $s$ to determine demand or supply. The shorter her predicted time to crash, the smaller the desired volume of trade. In addition, they demand the security depending on the predicted time. The price determined from the market-clearing condition:

$$\frac{1}{N}\sum_{\alpha\in A} e_{\alpha,k+1}(S_{k+1}) = -\bar{X}_k \ln\frac{S_{k+1}}{S_k} + r\{(1+v)\bar{X}_k - \bar{X}_{k+1}\} = 0$$

is obtained by $\ln S_{k+1} = \ln S_k + r\left(1 + v - \frac{\bar{X}_{k+1}}{\bar{X}_k}\right)$. If $k < T_c$, since the average predicted remaining time to crash over all agent is: $\bar{X}_{k+1} = \bar{X}_k - q + p\bar{X}_k\bar{\epsilon}_{k+1}$,
where $\bar{\epsilon}_{k+1} = \sum_{\alpha\in A}\epsilon_{\alpha,k+1}/N$, the logarithmic price satisfies:

$$\ln S_{k+1} - \ln S_k = r\left(v + \frac{q}{\bar{X}_k} - p\bar{\epsilon}_{k+1}\right).$$

Therefore, comparing this equation with Eq. (2), this discrete-time model can be regarded as a discrete analogue of the continuous model in the previous section.

## 4. Agent-based simulation

In this section, we perform an agent-based simulation based on the micro-foundation of the previous section. In this simulation, each agent's predicted remaining time to crash is generated individually. The simulation procedure is as follows. For $k = 0,1,2,\cdots$,

- Generate $N$ independent random variables $\epsilon_{\alpha,k+1}, \alpha \in A$ from the standard normal distribution $\mathrm{N}(0,1)$.
- Create $X_{\alpha,k+1} = X_{\alpha,k} - q + p\bar{X}_k\epsilon_{\alpha,k+1}$ for all $\alpha \in A$.
- If $\sum_{\alpha\in A} X_{\alpha,k+1} > 0$:
    - Set $\bar{X}_{k+1} = \sum_{\alpha\in A} X_{\alpha,k+1}$.
    - Compute $\ln S_{k+1} = \ln S_k + r\left(1 + v - \frac{\bar{X}_{k+1}}{\bar{X}_k}\right)$.

- Return to the first step.

- Else:
  - Set $\bar{X}_{k+1} = 0$.
  - Compute $\ln S_{k+1} = \ln S_k + r(1 + v)$.
  - Break.

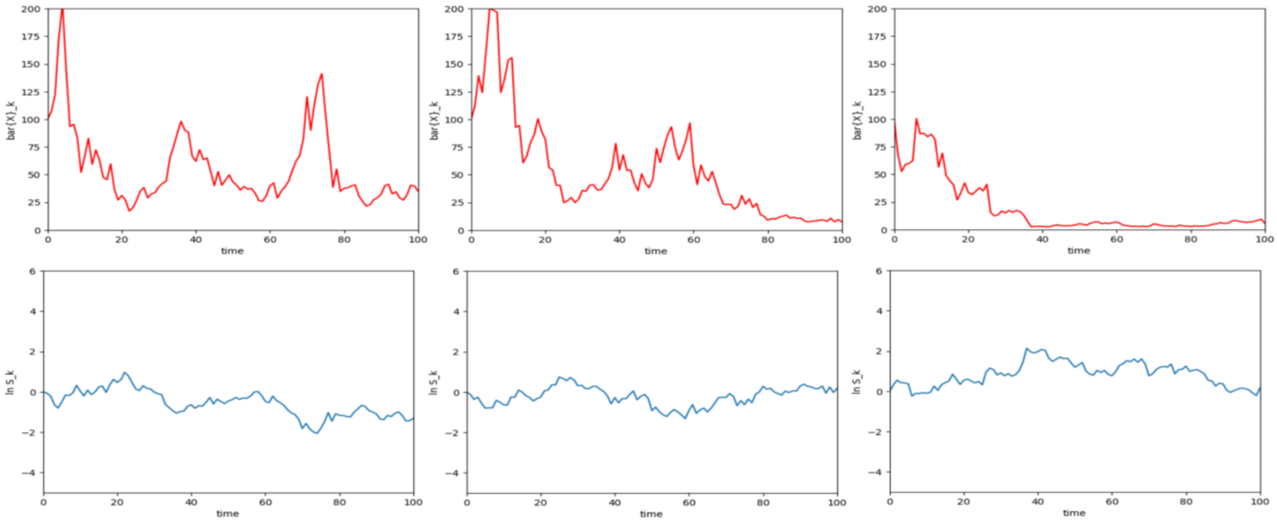
Here, the simulation is performed with the following parameter values:

$$N = 1000, \quad q = 0.1, \quad p = 7, \quad r = 1, \quad v = 10^{-5}$$

and $X_{\alpha,0} = 100$ for all $\alpha \in A$ for one hundred time steps.

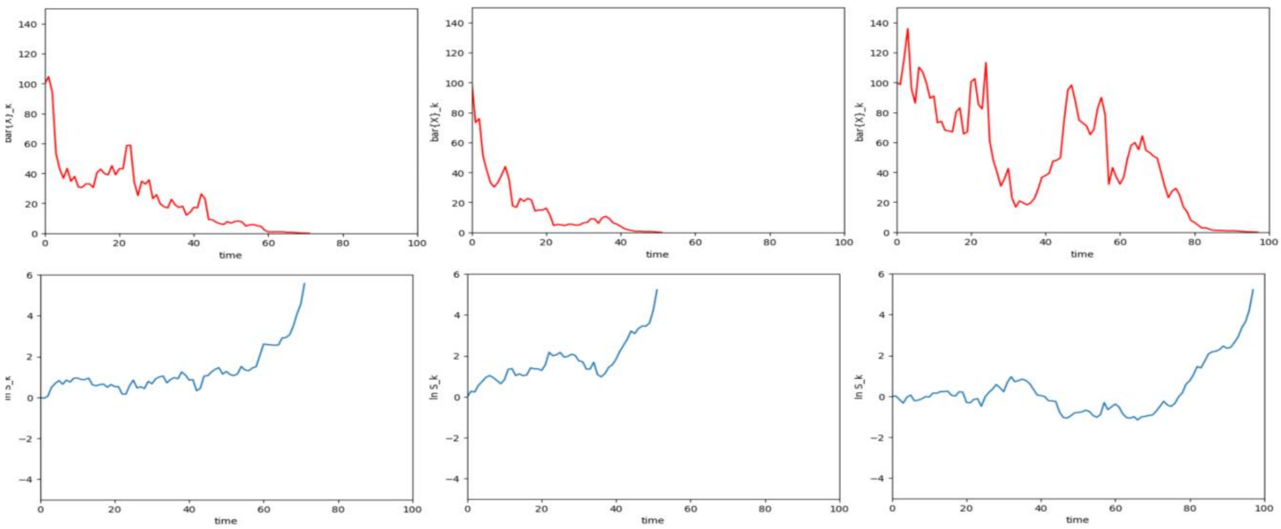
Figure 1 and Figure show three typical simulation results for the cases where the average predicted period does not reach zero and reaches zero in finite time, respectively.

*Figure 1.* Average predicted time to crash and the price movements without bubble




*Note:* The red lines (upper three panels) represent typical simulation results of the average predicted period $\bar{X}_k$ and the blue lines (lower three panels) the log price $\ln S_k$. It can be observed that if the average predicted period does not reach zero over a long time, there is no sudden bubble occurs.

*Figure 2.* Average predicted time to crash and the price movements with bubble




*Note:* The red lines (upper three panels) represent typical simulation results of the average predicted period $\bar{X}_k$ and the blue lines (lower three panels) the log price $\ln S_k$. When the average predicted period becomes zero in finite time, a sharp price increase can be observed immediately before the crash.

The red lines (upper three panels) in each figure represent the average predicted period $\overline{X}_k$ and the blue lines (lower three panels) the log price $\ln S_k$. When the average predicted period does not reach zero over a long time, there is no sudden bubble as shown in Figure. On the other hand, when the average predicted period becomes zero in finite time, a sharp price increase can be observed immediately before the crash, as shown in Figure. Therefore, it can be said that the excess demand model given in the previous section provides a somewhat adequate micro-foundation for the super-exponential and finite-time singular bubble.

## 5. Conclusion

In this paper, we have proposed a model of security price with bubble in which prices diverge and crash in finite time. Just before the bubble crashes, the price increases super-exponentially. As in existing studies that have presented such models, the agents' investment strategies incorporate the predicted time until the crash, but the continuous-time price model derived is simpler than before due to the predicted time newly modeled in this paper. In addition, a discrete-time excess demand model has been proposed to provide a micro-foundation of the continuous-time model. The price is determined such that the average of agents' excess demand is balanced to zero. The discrete-time price model derived in this way has similar characteristics to the continuous-time price model developed earlier and can be considered to represent finite-time singularity. Then, we have performed an agent-based simulation based on the excess demand model to examine the price behavior. As expected, we confirmed that prices sometimes diverge and increase super-exponentially within a finite time period.

In the future, we would like to create price models that can reproduce the log periodicity of the bubble curve of Sornette et al. (2013) in addition to the finite-time singularity with appropriate micro-foundations. In addition, we would like to consider other types of micro-foundations than the excess demand model. In particular, we would like to investigate the effect of agents' limit orders on the bubble curve of Sornette et al. (2013) by creating some suitable limit order book models of security trades.

We would also like to make a comparison with many other bubble models that model the supply and demand for securities. Among them, in Bouchaud and Cont (1998), a phase transition between a bubble and its crash is observed in a simple excess demand model developed by them. It would be interesting to investigate the difference between the state immediately before their phase transition and the finite-time singular bubble observed in the agent-based model of this paper.

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