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Mechanical anisotropy of a two-phase composite consisting of aligned elliptical inclusions

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Abstract: The shape preferred orientation of rock constituents results in an overall mechanical anisotropy. A differential effective medium (DEM) type of scheme predicting the effective anisotropic viscosity of a composite consisting of aligned elliptical inclusions is proposed and validated by finite element modeling. A comparison with an existing self-consistent averaging (SCA) scheme is given and the DEM scheme is shown to provide an improved estimate of the effective normal and shear viscosity for high inclusion concentrations.

Keywords: effective anisotropic viscosity, elliptical inclusions, differential effective medium, finite element modeling.

It has been recognized that the overall mechanical response of a heterogeneous rock may become anisotropic due to the development of shape preferred orientation (SPO). In the limiting case, a laminate of equal phase abundances exhibits a maximal degree of the anisotropy, where shear and normal viscosities yield values corresponding to the lower (Reuss) and upper (Voigt) theoretical bounds. The model of a laminate is not suitable for studying the transient stage of the anisotropy evolution during the SPO build up. Another analytical estimates of the overall anisotropy incorporating a finite magnitude of SPO was proposed (Fletcher, 2004). The model builds on an analytical solution for an elliptical inclusion embedded in an anisotropic matrix (Willis, 1964) and the effective normal and shear viscosities are evaluated using the self-consistent averaging technique. In this work, we derive and numerically validate an overall anisotropic viscosity estimate based on the differential effective medium approach.

Methods

Analytical model

The effective normal μ_n^{eff} and shear μ_s^{eff} viscosities are defined as the ratios of the appropriate components of the spatial averages of the deviatoric and strain rate. The fundamental relation between inclusion μ^{incl} and host μ^{host} viscosity, inclusion concentration f and effective viscosity μ^{eff} is given by (Nemat-Nasser and Hori, 1993):

$$\left(\mu_{ijkl}^{host} - \mu_{ijkl}^{eff}\right)\langle\dot{\varepsilon}_{kl}\rangle = f\left(\mu_{ijkl}^{host} - \mu_{ijkl}^{incl}\right)\langle\dot{\varepsilon}_{kl}^{incl}\rangle \qquad (1).$$

In the dilute limit, the strain rate average in the inclusion phase may be approximated by the strain rate of an isolated inclusion embedded in an isotropic host. The strain rate within the inclusion is uniform and given by (Willis, 1964):

$$\frac{\dot{\varepsilon}_{xx}^{incl}}{\dot{\varepsilon}_{xx}^{\infty}} = \frac{\sqrt{\delta}+\eta}{R_n\sqrt{\delta}+\eta}, \frac{\dot{\varepsilon}_{xy}^{incl}}{\dot{\varepsilon}_{xy}^{\infty}} = \frac{\sqrt{\delta}+\eta}{\sqrt{\delta}+\eta R_s} \quad (2),$$

where $\eta=1/2(\sigma+\sigma^{-1})$ is the shape factor, σ is the aspect ratio of the inclusion, $\delta=\mu_n^{host}/\mu_s^{host}$ is the anisotropy factor and $R_n=\mu^{incl}/\mu_n^{host}$, $R_s=\mu^{incl}/\mu_s^{host}$ are the inclusion-host viscosity ratios. Hence, the normal and shear viscosity of a dilute composite consisting of aligned inclusions can be estimated using the isotropic limit of (2) in (1):

$$\frac{\dot{\varepsilon}_{xx}^{incl}}{\dot{\varepsilon}_{xx}^{\infty}} = \frac{1+\eta}{R+\eta}, \frac{\dot{\varepsilon}_{xy}^{incl}}{\dot{\varepsilon}_{xy}^{\infty}} = \frac{1+\eta}{1+\eta R} \quad (3),$$

where $R = \mu^{incl}/\mu^{host}$ is the ratio of the inclusion and host viscosity.

Self-consistent averaging (SCA) has been applied to a composite consisting of aligned ellipses by Treagus (2003) and Fletcher (2004). In this approach, the viscosity ratios in (3) are evaluated with respect to the effective medium and not the original host material. In Fletcher's work, the inclusion strain rate has been evaluated according to (2) taking into account the overall anisotropy. The effective normal viscosity normalized by the host viscosity $\beta_n=\mu_n^{eff}/\mu^{host}$ and the effective anisotropy factor δ^{eff} are determined by solving a system of non-linear equations:

$$\begin{aligned}&\eta\beta_n^2+\{\sqrt{\delta^{eff}}[f+(1\text{-}f)R]-\eta[(1\text{-}f)+fR]\}\beta_n-\sqrt{\delta^{eff}}\,R=0\\&\beta_n^2+\{\eta\sqrt{\delta^{eff}}[f+(1\text{-}f)R]-\delta^{eff}[(1\text{-}f)+fR]\}\beta_n-\eta(\sqrt{\delta^{eff}})^3\,R=0\end{aligned} \quad (4),$$

In the differential effective medium (DEM) another approach is taken to estimate overall properties at high concentration. Here, the medium is constructed in an iterative manner by placing individual inclusions into the host and reevaluating the host properties afterwards (e.g. Berryman *et al.* 2002). We use a DEM scheme that takes into account the anisotropy according to (2). The effective viscosities are obtained by integrating two coupled ordinary differential equations:

$$\begin{aligned}\frac{d\beta_n}{df}&=\frac{1}{1\text{-}f}(R\text{-}\beta_n)\frac{\sqrt{\delta^{eff}}+\eta}{\sqrt{\delta^{eff}}\,R/\beta_n+\eta}\\\frac{d\beta_s}{df}&=\frac{1}{1\text{-}f}(R\text{-}\beta_s)\frac{\sqrt{\delta^{eff}}+\eta}{\sqrt{\delta^{eff}}+\eta R/\beta_s}\end{aligned} \quad (5),$$

where $\beta_s=\mu_s^{eff}/\mu^{host}$ and the initial values of both β_n and β_s are 1.

Finite element modelling

To validate the SCA and DEM schemes, a finite element model (FEM) that allows us to directly resolve the mechanical response of composites consisting of numerous inclusions of a constant size and orientation is employed. A large number of inclusions is needed for a sufficient representation of the composite. This results in a high discretization level in our models with resolutions that exceed one million degrees of freedom. In this study we utilize our optimized unstructured mesh FEM code MILAMIN implemented entirely in MATLAB (Dabrowski *et al.*, 2008). We systematically scan through the parameter space of inclusion concentration (up to 50%), aspect ratio (up to 16) and viscosity ratio (between 1/1000 and 1000). Non-overlapping inclusions of equal size are seeded randomly in the computational domain. The effective normal and shear viscosity are measured by applying corresponding kinematic boundary conditions.

Results

Numerical and analytical results for models consisting of elliptical inclusions are presented in figure 1. It is evident in figure 1a that the presence of the strong host results in a higher effective viscosity for composites of equal phase concentrations. The scatter of the effective viscosity due to changing the composite configuration increases with the inclusion concentration. However, even in densely packed cases, the spread is rather small and the data presented at 5% concentration increments overlap minimally in terms of effective viscosity. The impact of the viscosity ratio on the effective viscosity is depicted in figure 1b for models with 40% of inclusions. The effective viscosity is virtually insensitive to inclusion or host viscosity changes once the strong to weak phase viscosity ratio exceeds several hundred. The SCA provides a good estimate of the effective viscosity for inclusion concentrations below thirty percent. The SCA fails to predict the saturation effect for high concentrations and towards large viscosity ratios. The DEM estimate captures the effective viscosity saturation and provides a good fit to the numerical results.

Numerical simulations show that the effective normal viscosity is systematically greater than the shear viscosity (Fig. 1c). Both components saturate with respect to the viscosity ratio changes (Fig. 1d).

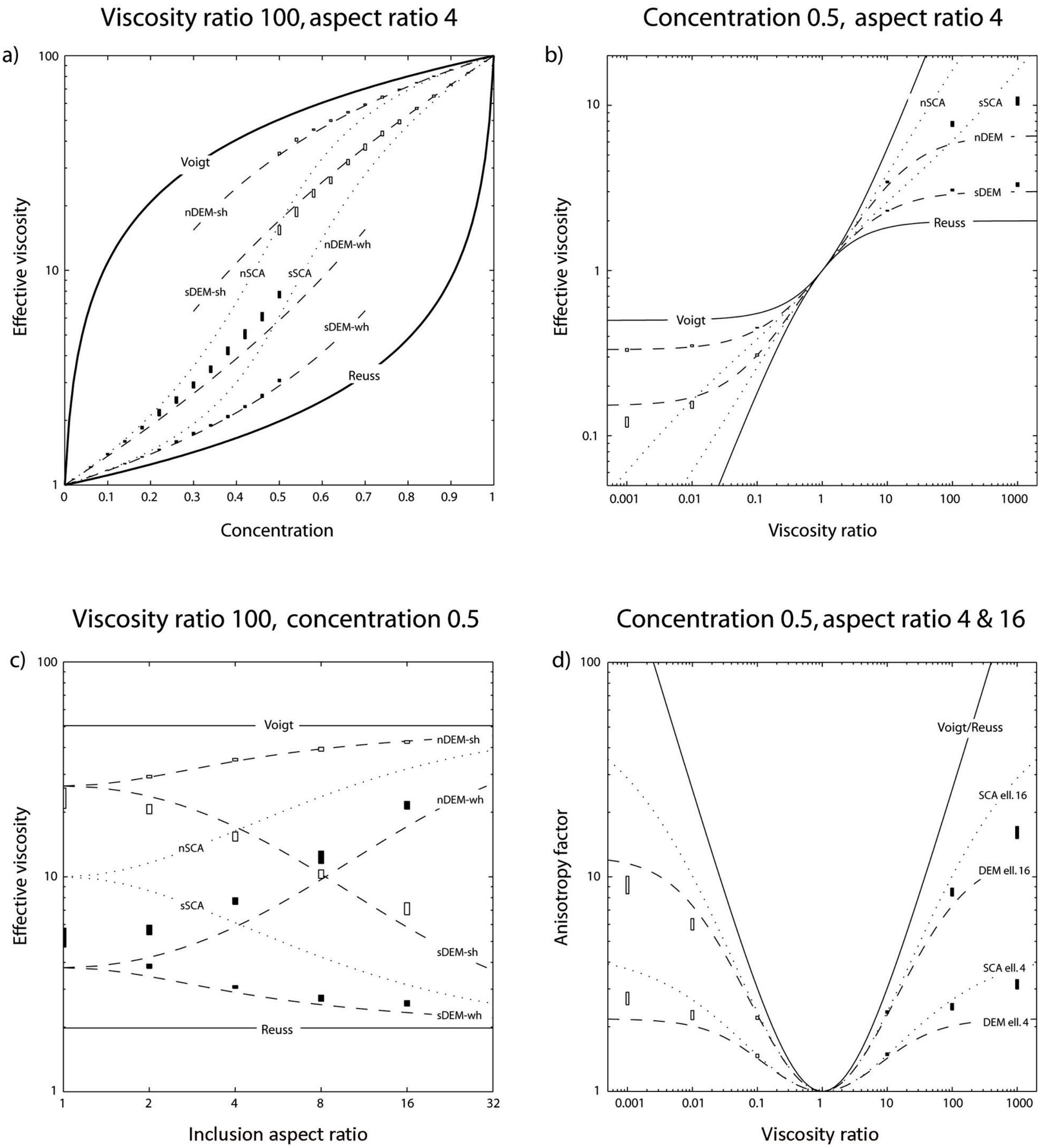

Figure 1. Effective viscosity of composites consisting of 256 non-overlapping inclusions. Open and filled bars correspond to models with weak and strong inclusion, respectively. Bottom and top of the bars are given by minimal and maximal values recorded for 10 samples. Upper (Voigt) and lower (Reuss) bounds, self-consistent average (SCA) and differential effective medium estimate for strong (DEM-sh, nDEM-sh, sDEM-sh) and weak host (DEM-wh, nDEM-wh, sDEM-wh) are given. Prefixes n and s indicate normal and shear viscosity; (a) viscosity ratio is set to 100 and concentration refers to the strong phase, (b) concentration is fixed at 40% and inclusion-host viscosity ratio is varied, (c) viscosity ratio is set to 100, aspect ratio set to 4 and concentration refers to the strong phase, (d) concentration is fixed at 40%, aspect ratio set to 4 and inclusion-host viscosity ratio is varied.

The effective viscosities respect the theoretical upper and lower bounds in all simulation runs. The DEM provides a good fit to the numerical data over the whole range of concentrations, whereas the quality of the SCA estimate deteriorates for densely packed composites. The normal viscosity in the weak case (or the shear viscosity in the strong case) is particularly precisely estimated.

Discussion

The DEM model provides better estimates over the SCA for high concentrations. The discrepancies between the scheme predictions reflect a fundamental difference between the two methods: the DEM is designed for inclusion-host systems, whereas the SCA is more suitable for a poly-grain medium, where none of the phases can be considered as inclusions.

This statement is corroborated by the FEM results that are predicted by the DEM scheme with a high accuracy up to large concentrations irrespective of the inclusion aspect ratio.

Our FEM results show that choosing a weak or strong phase for the inclusions leads to a significantly different effective response. This is in agreement with the notion of the load-bearing framework that was previously suggested in the geological literature (e.g. Handy, 1990). Existing theoretical models often have difficulties in predicting the overall viscosity in this transitional regime between the weak and strong phase supported structures (Ji *et al.*, 2001). The DEM scheme is well capable of differentiating the weak and strong phase supported structures and we suspect it may explain experimental data both in isotropic and anisotropic case.

Conclusions

A scheme based on the differential effective medium approach predicting mechanical properties of a composite consisting of aligned elliptical inclusions is proposed and numerically validated. The scheme is free of phenomenological parameters and provides a good fit to the numerical results in a wide range of inclusion concentration, aspect ratio and inclusion-host ratio. The weak and strong supporting phase cases are distinguished by the model leading to an improved estimate for compositions higher than 20 percent in comparison with the SCA. The anisotropic DEM scheme removes the deficiency of an unbound prediction in the limit of extremal viscosity ratios for high concentrations.

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