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Lyapunov method: a tool to describe fabric attractor in non-linear and heterogeneous flows with application to shear zones

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Abstract: In this contribution we review and expand the concept of attractor to non-linear flow system, introducing the concept of stability analysis to unravel some simple heterogeneous flow system. A semi-quantitative tool (using Lyapunov method) is suggested to predict the fabric attractor within some simple and general shear zones defined by non-linear flow tensor. The theory is briefly explained and it is shown the potentiality of this approach on describing the flow in heterogeneous shear zones. An example is discussed and a kinematic classification of some possible non-linear flow systems, using Lyapunov exponent analysis, is proposed.

Keywords: non-linear flow kinematics, heterogeneous shear zones, fabric attractors.

Deformation is generally treated as homogeneous and steady state (i.e. the kinematics of the flow at a given material particle is not varying with time and space) because the mathematical description becomes too complex otherwise. However, several field, experiment and theoretical oriented works showed that strain rate, as well as rheological properties of rocks, generally change during the deformation history. This implies that the deformation could vary non-linearly along space and with time (Fossen and Tikoff, 1997; Jiang and Williams, 1999; Trepmann and Stockhert, 2003). Non-linear flow implies that the principal asymptotically stable directions of flow, that behave as attractor or repulsor (Ruelle, 1981; Tabor, 1989), are expected to control the final orientations of the principal strain axes as well as the final fabric distribution, that could vary in time and in space. Heterogeneous deformation represents a classical example of autonomous non-linear system as it implies that the strain varies following a non-linear function (Ramsay, 1980), e.g. showing a vorticity or strain gradient along a specific direction (Jiang and Williams, 1999). Solutions of differential equations describing non-linear autonomous system are not obvious and if exist are not deterministic. These non-linear properties strongly limit any attempt to reconstruct the flow history in a unique way as different kinematics histories could produce the same results. As a consequence, geological structures cannot be generally described in terms of unique flow pattern. The aim of this extended abstract is to contribute to the knowledge of the possible flow pattern and related structures produced in some non-linear flow in order to understand or predict similarly to the linear case if some fabric attractors (McKenzie, 1979; Passchier, 1997) could be expected. With this purpose, we introduce the concept of stability analysis (using Lyapunov method) as a criterion to describe the flow pattern where the flow path cannot be derived by simple integration, and we discuss the properties of some non-linear flow that could be analysed with this method.

Method

Dynamical systems are often described in terms of differential equations. To determine the behaviour for longer periods it is necessary to integrate the differential equations, either through analytical means or through iteration, often with the aid of computers. If the dynamical system is a continuous time system it is called flow system. The classical differential equations defined by the flow tensor matrix L_{ij}, and used in strain analysis to describe flow rocks, are expressed as:

$$\dot{X} = L_{ij} X \qquad (1).$$

To solve equation (1), the system needs to be integrated and the obtained integrated function must be inverted. These two conditions are not obvious or straightforward even in simple one dimensional case. Its solution defines the flow path equation necessary to obtain the displacement path and the finite strain ellipse. If the tensor L_{ij} is defined by a constant parameter, the flow tensor defines a linear dynamical system. If L_{ij} is defined by a non-constant parameter as linear and non-linear function F(x), the solution is not always easy to unravel and does not necessarily exist (Tabor, 1989). If L_{ij} is time dependent the flow is non-steady. The main point we wish to stress is that, sometimes, it is possible to estimate in a qualitative way the properties of the solutions and their behaviour for $t \rightarrow \infty$, understanding if the differential equations bear an attractor or not, without solving the non-linear differential equation, and by simply studying the properties of functions that approximate the flow tensor. Some of these functions are the Lyapunov functions (Tabor, 1989). Depending on the properties of the non-linear system, different methods of constructing Lyapunov functions need to be applied. Hereafter, we will briefly describe what the Lyapunov functions are and how to build a Lyapunov function *ad hoc* for a non-linear flow system.

Construction of Lyapunov function for non-linear systems

Lyapunov functions are by definition functions which prove the stability of a fixed point in dynamical system or non-linear autonomous differential equations. In 3D, a fixed point X is a point for which:

$$\dot{X} = L_{ij} X = 0 \qquad (2).$$

Stability defined by a fixed point corresponds nicely with the intuitive notion (not rigorous, but this is not the point here) that a fixed point is stable when a starting condition close to the fixed point remains close for all time. The notion of asymptotic stability is a little stronger than stability concept and holds for a fixed point when nearby conditions not only remain close for all time but actually move toward the fixed point approaching it asymptotically. This corresponds to the concept of attractors used in linear differential equations (Tabor, 1989) and rock flow analysis (Passchier, 1997). Given the previous stability concept, a pratical definition of Lyapunov function could be done as follows. Let be x an isolated fixed point of the system (2) and located in some open set D in the phase space A of (2). Then, V(x), a differential function mapping from the phase space A to the real line R is a Lyapunov function if it has the following properties: a) $V(x) \geq 0$ $\forall\, x \in D$; b) $V(x) = 0 \Leftrightarrow x = x'$; and c) the derivative of V(x) with respect to t along a solution curves is non positive for all t, i.e. $dV(x(t))/dt < 0$.

If a Lyapunov function V(x) exists, then x' is stable. If the x' is the only point in D for which $dV(x)/dt = 0$, then x is asymptotically stable. In a few words, the procedure states that if we have some functions V(x), defined over the phase plane near the fixed point we are interested in, then we need picking an initial condition and let the dynamic system defined in equation (1) evolve moving around on the surface described by this function. Suppose this function has only one minimum, located at the fixed point x. If we start at some initial condition near x we will observe a trajectory on the surface of our function. If that trajectory leads to decreasing value of V(x) then we know that we must approach the minimum of V(x), i.e. the fixed point. The existence of V(x) with all its stated properties guarantees the stability of x. The main limit of this approach is that there is not a general method to construct a Lyapunov function and the inability to find a Lyapunov function is inconclusive with respect to stability. The theorem merely tells us what happen if we can construct one. On the other hand, the Lyapunov function is never unique as we can readily imagine multiplying or adding a costant to V(x) without changing the essence if the properties noted above. Let's show and example with a simple heterogeneous flow tensor. Consider the non linear flow tensor defined in this way:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -f(x,y) & a \\ a & -f(x,y) \end{pmatrix} \qquad (3).$$

This differential equation represents a flow tensor having a gradient along the pure shear direction inducing a heterogeneous deformation. This function f(x,y), that defines the gradient, could describe both dilatant as well as constant area flow. Angular velocities are fixed. In this case, to unravel the strain rate tensor and the vorticity tensor it is necessary to apply the relation:

$$L_{ij} = ½(L_{ij} + L_{ji}) + ½(L_{ij} - L_{ji}) \qquad (4).$$

The first term defines the strain rate tensor, the second one defines the angular velocity tensor (McKenzie, 1979). This flow tensor has a solution in (0,0). However, its solution depends on the values of the dissipative terms f(x,y) and the myriad possible solutions could not always be unravelled by simple integration. So let's define a Lyapunov function L to determine the stability of the equilibrium solution in (0,0).

Let be $L=x^2+y^2$, we have $\partial L/\partial t=2x\dot{x}(t)+2y\dot{y}(t)$, from equation (3) we obtain:

$$\frac{\partial L}{\partial t} = -f(x,y)(2x^2 + 2y^2) \qquad (5).$$

1) If f(x,y) is positive semidefinite (does mean $f(0,0) = 0$ and $f(x,y) \geq 0$ for $x,y \neq 0$) then $\partial L/\partial t \leq 0$ so the flow path is uniformily stable and have an attractor.

2) If f(x,y) is positive definite (does mean $f(0,0) = 0$ and $f(x,y) > 0$ for $x,y \neq 0$) then $\partial L/\partial t < 0$. It means that whatever is the displacement path, the flow is asymptotically stable and after large strain accumulation it develops a stable attractor. In this case, the orientation of the finite strain ellispoid is controlled by an attractor.

3) If f(x,y) is negative definite (does mean $f(0,0) = 0$ and $f(x,y) > 0$ for $x,y \neq 0$) then $\partial L/\partial t > 0$. It means that whatever is the displacement path, the flow is unstable, exploding for large strain accumulation. In this case, the dilatant system is highly dissipative and no stable attractor is expected.

Summarizing, using the Lyapunov function, a simple planar heterogeneous tensor has been analysed and the properties of the gradient functions to obtain final fabric attractors has been described. To obtain an heterogenous flow tensor with stable direction we have to define some reasonable gradient function f(x,y) positively definite.

Lyapunov exponent: indirect method to test asymptotic behaviour

If a Lyapunov function cannot be constructed, an alternative method to test the possible existence of a fabric attractor and to understand what kind of attractor it should be expected is the indirect Lyapunov method, the one numerically developed by Wolf *et al.* (1985). The Lyapunov method is classically used as a tool to unravel possible chaotic motion but could be introduced as well to describe trajectory of ordinary differential equations and to test their behaviour for $t \rightarrow \infty$ without finding a complete solution of the system. Basically, this method gives a qualitative degree of divergence of orbits or deformation path in non-linear dynamical system from their principal eigenflows by finding and studying the Lyapunov exponent. Hereafter, a brief analytic description of the method will be proposed to test if the non-linear function could have a fixed attractor or stable direction.

Let have a one-dimensional system defined by:

$$\dot{x} = f(x) \qquad (6).$$

Suppose we have two starting conditions, x_a and x_b, then the trajectories deriving from such two initial conditions are: $\dot{x}_a = f(x_a)$ the first trajectory, and $\dot{x}_b = f(x_b)$ the second trajectory.

$$\dot{d} = \dot{x}_a - \dot{x}_b \qquad (7),$$

is the difference between these two trajectories.

Equation (7) could be rewritten in a more operative way as follows:

$$\dot{d} = \dot{x}_a - \dot{x}_b = f(x_a) - f(x_b) \approx f'(x_a)d \qquad (8).$$

Here, it has been assumed that d is small (infinitesimal) and $f(x_b)$ could be expanded into a Taylor series.

Assuming that $f'(x_a)$ is approximately constant, we have that the distance changes exponentially in this way:

$$D(t) = d_0 e^{\lambda(t-t_0)} \qquad (9),$$

where d_0 is the initial distance between the trajectory and t_0 is the initial moment of time. Rewriting equation (9) we have:

$$\lambda = \frac{1}{t - t_0} \ln \frac{D(t)}{d_0} \qquad (10).$$

If the system is linear, λ is constant and if it does not depend on the initial condition, d has to be zero because the two trajectory in equation (8) show the same path.

For non-linear and non-autonomous system, λ is not constant and this implies that equation (10) has to be defined as a mean divergence of the two trajectories at large t, finding the divergence values as a limit for t going to infinity.

The limit is:

$$\lambda = \lim_{t \to \infty} \frac{1}{t - t_0} \ln \frac{D(t)}{d_0} \quad (11),$$

and λ is called Lyapunov exponent. In a non-linear system, the gradient of the function could change throughout the all trajectory being time dependent. As a consequence, it is necessary to find an averaged Lyapunov exponent of the whole trajectory. In practice, the Lyapunov exponent should be redefined in this way:

$$\lambda = \lim_{t \to \infty} \frac{1}{t_n - t_0} \sum_{k=1}^{n} \ln \frac{D(t_k)}{d_{tk-1}} \quad (12).$$

As showed graphically in figure 1, computing the different trajectories (thinnest line in figure 1) at different initial point (if time dependent), the method consists on describing how much such trajectory computed at different point could diverge from a reference trajectory (thickest black curved line in figure 1) initially chosen. In this context, the reference trajectory could be a numerically calculated eigenflow. For an n-dimensional continuous system, the function f' has to be replaced by the Jacobian matrix J = Grad f.

Doing the same reasoning made in equations (6-10), the n Lyapunov exponent calculated along the eigenvectors of an n-dimensional dynamic system could be syntethically described in this way:

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t_n - t_0} \sum_{k=1}^{n} \ln \frac{J_{ij}(xi)}{d_{tk-1}} \cong$$
$$\cong \lim_{t \to \infty} \frac{1}{t_n - t_0} \sum_{k=1}^{n} \ln \mu_i(t) \quad (13),$$

being μ_i the eigenvalues of the jacobian matrix. The set of λ_i forms the Lyapunov spectrum.

Now, applying such method to a 3D flow system, where the Jacobian matrix could represent the strain rate matrix, time dependent or a non-linearly time independent matrix, three Lyapunov exponent λ_i (λ_1, λ_2, λ_3) are expected. Intuitively assuming the initial starting point at time t_0 as a sphere field (assuming the Lyapunov at this point as zero values) if the Lyapunov exponent calculated at different point and time t_i on the reference function is not zero, but there exist almost one positive value, the initial sphere become deformed as an ellipsoid (Fig. 2) an the principal axes of such ellipsoid defines the biggest Lyapunov exponent. Equation (13) indicates a strict relation between the Lyapunov exponent and the eigenvalues of the dynamical system, and states that by a simple calculation of the eigenvalues at different time t_i, it could be understood if the non-linear component of the system behave chaotically or not.

Pratical plan to finding Lyapunov exponent

a) Assuming that we have a system $\dot{x} = f(x)$, find the basic trajectory. To do that, we should integrate numerically the system for a sufficiently long period of time. b) Then we find the Jacobian J of our system $\dot{\eta} = J\eta$. c) Fix the initial deviation from the trajectory at some values. d) Integrate the jacobian during a time $t = t_k - t_{k-1}$.e)

Find the divergence: $\dfrac{d(t_k)}{d(t_{k-1})} = \left| \dfrac{\eta(t, t_{k-1})}{\eta(t_{k-1})} \right|$. f) Start the

next time step in the direction $\eta(0) = \left| \dfrac{\eta(t, t_{k-1})}{\eta(t_{k-1})} \right|$ and

go to step c) until the end of trajectory. g) Finally, find the Lyapunov exponent using equation (13). See Wolf *et al.* (1985) for a detailed description of the method proposed here.

Result: qualitative description of asymptotic solution

Flow tensor can reasonably describe strain distribution within shear zones (Ramberg, 1975; McKenzie, 1979), and as finite strain ellipsoid is controlled by the attractors of the flow tensor, then for large strain accumulation, fabric distribution could be reasonably predicted by mean of a stability analysis (Passchier, 1997). This implies that if we can find Lyapunov function to describe stability and asymptotic behaviour of non-linear flow tensor, then we can predict the asymptotic behaviour of strain distribution within high strain heterogeneous shear zones defined by non linear flow tensor, without the necessity of solving the flow equation. An example has been described and discussed. Moreover, using

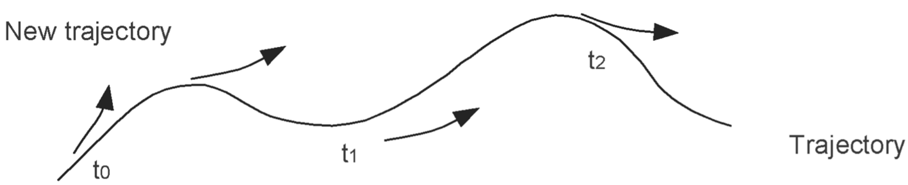

Figure 1. Lyapunov exponent of a trajectory. The thickest line with no arrow, defines the principal trajectory chosen. The thinnest lines with arrows define the computed different trajectories starting from different points

Lyapunov exponent analysis some end member solution could be expected for general non linear flow system:

a) If the system is dissipative, implying a volume contracting (Tr $J_{ij} < 0$), the sum of the three Lyapunov exponent is negative, implying that at least one of the λ_i is negative. If the biggest eigenvalues (as norm calculated) is negative, it should be expected that for $t \rightarrow \infty$ the dynamic system is controlled by a stable attractor. The same should be expected if the system is a conservative system (isochoric flow Tr $J_{ij} = 0$).

b) If the biggest Lyapunov exponent λ_1 is positive, the system is chaotic. If not, the following possibilities are expected: i) if $\lambda_1 = 0$ and $\lambda_2 > \lambda_3 < 0$ we have a limit cycle; and ii) if $\lambda_1 = \lambda_2 = 0, > \lambda_3 < 0$ we have a torus.

If the system is characterized by an expanding volume change (as an expanding shear zone), theory says that the attractor expected should be chaotic and not fixed or asymptotically stable (Fig. 3).

Applying these concepts to heterogeneous high-strain shear zones, if the heterogeneous shear zone is a confined and/or non-linearly contracting shear zone, a stable attractor should be expected after a large accumulation of strain (Fig. 4). If the shear zone is dilatant, a possible chaotic behaviour should be expected. This last consideration seems to be in conflict with the fact that chaotic fabric distribution has been rarely described in dilatant high-strain shear zones intruded by partial melting. However, to better understand the effective validity of these end members within real physical system a test involving mechanical parameters should be performed in the next future.

Conclusion

With this contribution we describe a semiquantitative tool using Lyapunov functions, which can be used to predict fabric distribution within heterogeneous shear zones without being forced to solve complex analytical differential equations. We discussed a simple class of non-linear flow tensor. Finally we described the possible flow path expected within simple heterogenous planar shear zones.

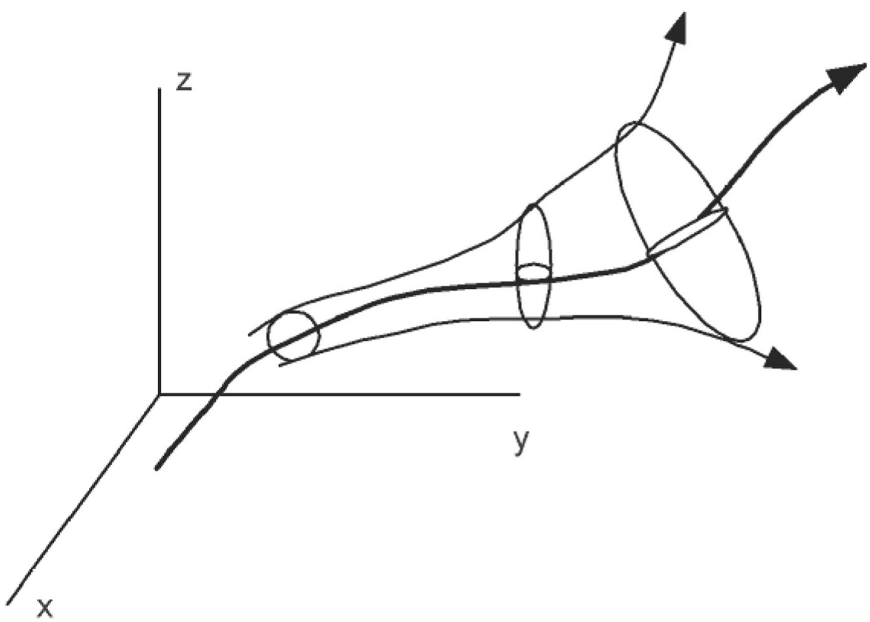

Figure 2. Example of 3D chaotic function (thickest black line). The small sphere indicates the initial conditions at t_i. The different ellipses indicate the rate of divergence expected for different trajectories starting at different points along the function. The principal axes of each ellipse define the three Lyapunov exponents. In this case, the biggest Lyapunov exponent is positive and indicates a divergent chaotic flow pattern.

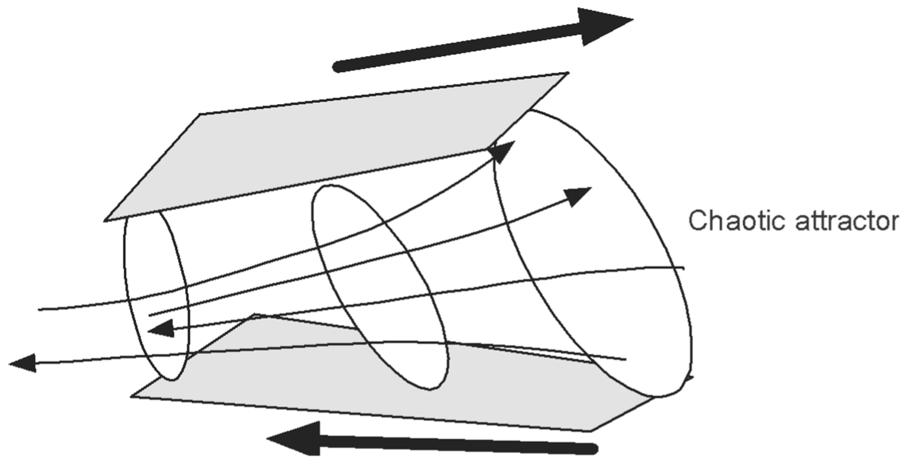

Figure 3. Expanding dilatant shear zone. Black arrows show the expected deformation paths. Gray surfaces define the boundary walls and the ellipses the different averaged Lyapunov ellipses.

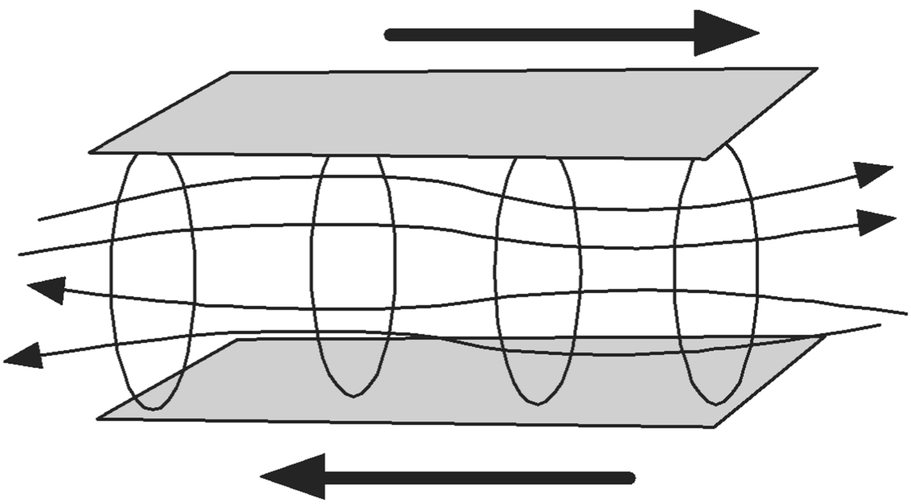

Figure 4. Conservative shear zone with non-linear but stable deformation path (black arrows). Gray surfaces define the boundary walls and the ellipses the different Lyapunov ellipses.

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