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Some kinematic properties of complex eigenvalues in 3D homogeneous flows

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Abstract: A mathematical investigation on some kinematic properties of 3D homogeneous flows defined by complex eigenvalues is presented. We demonstrate by mean of simple algebra analysis, that in a 3D flow system a clear threshold between pulsating and non-pulsating fields does not exist. This implies that the existence of a stable or pulsating pattern in 3D flow is not simply imposed by the kinematic vorticity numbers. Moreover, we show theoretically that a 3D flow path having complex eigenvalues could evolve into a stable flow path. These results are applied to the kinematic analysis of some non-dilational and dilational monoclinic and triclinic flows.

Keywords: flow kinematics, deformation, ghostvector, pulsating pattern.

In homogeneous and steady state flow we can describe the kinematics of structures in terms of either velocity or deformation gradients. Applying these concepts to deformable rocks, we can study the progressive deformation and related deformation path pattern in 2D or simple 3D systems. In detail, steady state flow patterns are critically dependent on flow parameters like the relative magnitude of the vorticity number (W_n), dilatancy parameter (A_n) and strain rate. In line with this approach, Ramberg (1974) and McKenzie (1979) indicated for which vorticity and strain rate ratio several pulsating and non-pulsating strain paths can occur. Ramberg (1974) showed that for 2D deformation, the threshold limit between the oscillatory and non-oscillatory field is defined by the eigenvalues of the strain rate matrix. If the eigenvalues are all real numbers, the eigenflows give rise to exponential deformation paths and the eigenvectors behave or as attractors or as repulsor (Ruelle, 1981; Passchier, 1997). If the eigenvalues are purely imaginary, the eigenvectors do not behave neither as attractor or repulsor (Ramberg, 1974; Weijermars, 1993). A complete description of such pulsating strain in 2D flow systems based on analytical and experimental work was presented by Weijermars (1991, 1993, 1998) and Weijermars and Poliakov (1993). Some examples of 3D pulsating path and strain history were first described analytically by Weijermars (1997). However a complete analytical and geometrical analysis of 3D flow patterns controlled by real or complex eigenvalues as well as their relative kinematics meaning in deformed rocks is lacking in the literature. In this contribution, we introduce a complete algebraic discussion of general 3D flow in domains where complex eigenvalues can occur and we discuss their kinematical meanings and relevance in describing geological structures.

Methods

Following Ramberg (1974) and McKenzie (1979) the flow can be described with respect to a geographic reference system by the velocity gradient tensor L_{ij}. The velocity gradient tensor (or flow matrix) L_{ij} could be decomposed into a symmetric stretching tensor D_{ij} and an antisymmetric vorticity tensor W'_{ij} as follows:

$$L_{ij} = \tfrac{1}{2}(L_{ij} + L_{ji}) + \tfrac{1}{2}(L_{ij} - L_{ji}) = D_{ij} + W'_{ij} \quad (1).$$

The first term on the right, D_{ij}, is also called strain rate matrix, the second term, W'_{ij}, describes, in a steady state and homogeneous flow, the angular velocity of an orthogonal pair of material lines in the deformation medium with respect to a geographical reference system. Eigenvectors of the D_{ij} term represent the maximum, medium and minimum Instantaneous Streching Axes (hereafter called ISA) of the flow pattern. If L_{ij} does not change from point to point, the flow is homogeneous, otherwise it is heterogeneous. If L_{ij} does not vary with time, the flow is steady, otherwise it is not steady. As showed by Jiang (1994), heterogeneous flow is inevitably non steady. In this paper we will limit our analysis to steady and homogeneous flows without loss of generality. In a steady flow, ISA and eigenvector of L_{ij} represent directions that do not change orientation during the progressive deformation history. Now let's consider a 3D flow system in which we fix again our reference system to the ISA, ideally moving the external reference system to the internal ISA system. In this reference system, the spinning matrix defines exactly the non coaxial component and their eigenvalues are zero. Hereafter, the coordinate system x, y, z will be considered with respect to the ISA. Following Iacopini *et al.* (2007), a general flow matrix L_{ij} written with respect to the ISA can be simplified as:

$$L_{ij} = \begin{pmatrix} a & p & -q \\ -p & b & r \\ q & -r & c \end{pmatrix} \quad (2),$$

where p, q, r are the off-diagonal coefficient of the gradient velocity tensor (function of the angular velocity and shear angular values) and a, b, c represent the stretching rate components. This tensor can be monoclinic or triclinic depending on the off-diagonal components. The time-integration of the strain rate matrix (eq. 2) gives rise to the finite position gradient tensor at time t, accumulated by progressive deformation at constant invariable flow parameters (Ramberg, 1974), that rewritten respect to the ISA reference coordinate system, using the Tikoff and Fossen formulation (Tikoff and Fossen, 1993), give rise to the following displacement path equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 \begin{pmatrix} \nu_{1x} \\ \nu_{1y} \\ \nu_{1z} \end{pmatrix} e^{(\lambda_1)t} + C_2 \begin{pmatrix} \nu_{2x} \\ \nu_{2y} \\ \nu_{2z} \end{pmatrix} e^{(\lambda_2)t} + C_3 \begin{pmatrix} \nu_{3x} \\ \nu_{3y} \\ \nu_{3z} \end{pmatrix} e^{(\lambda_3)t} \quad (3),$$

where C_i, similarly to the 2D case, represents the coordinate coefficient, function of initial condition. To describe the flow pattern produced by equation (3) it is necessary to understand the properties of the eigenvalues λ_i and the column vectors ν_i. The ν_{ij} column vectors represent the correspondent eigenvectors of λ_i. In a steady flow, the eigenvectors ν_i control the main flow pattern defining the main attractors and repulsors if defined by real eigenvalues (Passchier, 1997). The eigenvectors defined by complex eigenvalues hereafter are called "ghostvectors". The eigenvalues λ_i of the matrix L_{ij} are obtained calculating the roots of the characteristic polynomial of L_{ij} which are:

$$\lambda^3 - \lambda^2 (\mathrm{Tr}\, L_{ij}) + \lambda (p^2 + q^2 + r^2 + ab + ac + bc) - \det L_{ij} = 0 \quad (4),$$

rewritten as:

$$x^3 + \nu x^2 + \chi x + k = 0 \quad (4a),$$

being:

$$\begin{gathered} \chi = p^2 + q^2 + r^2 + ab + ac + bc \\ \nu = -(\mathrm{Tr}\, L_{ij}) \\ k = -\det L_{ij} \\ j = \chi - \nu 2/3 \\ h = k + (\chi\, \nu)/3 - 2\nu 3/27 \end{gathered} \quad (4b),$$

In this case, the polynomial characteristic equation represents a third degree equation function of λ_i. To solve the third degree equation, we followed the so called Cardano method which is also described in several algebra textbooks (Birkhoff and Mac Lane, 1996). Posing $\lambda = m + n - \nu/3$, after several algebrical substitutions (see in detail analytical procedure in Iacopini *et al.*, 2007) we obtain the following relations:

$$\lambda_1 = m + n + \frac{TrL_{ij}}{3} \quad (5a),$$

$$\lambda_2 = m\left(\frac{-1+i\sqrt{3}}{2}\right) + n\left(\frac{-1-i\sqrt{3}}{2}\right) + \frac{TrL_{ij}}{3} \quad (5b),$$

$$\lambda_3 = m\left(\frac{-1-i\sqrt{3}}{2}\right) + n\left(\frac{-1+i\sqrt{3}}{2}\right) + \frac{TrL_{ij}}{3} \quad (5c),$$

$$\mathrm{Re}(\lambda_2) = \mathrm{Re}(\lambda_3) = -\tfrac{1}{2}(m+n) + \left(\frac{TrL_{ij}}{3}\right) \quad (5d),$$

being:

$$m = \sqrt[3]{\frac{-h}{2} - \sqrt{\Delta}} \quad (6a),$$

$$m = \sqrt[3]{\frac{-h}{2} + \sqrt{\Delta}} \qquad (6b),$$

and:

$$\sqrt{\Delta} = \sqrt{\frac{h^2}{4} + \frac{j^3}{27}} > \frac{|h|}{2} > 0 \qquad (6c).$$

From these relations and using flow path equations in equations (3) and (6), it can be easily inferred that the real part of λ_1 will behave as the controlling eigenvalue for $t \rightarrow \infty$ only if (m+n) > 0. If (m+n) < 0 the real parts of the λ_2 solution are larger than λ_1 so they will dominate at $t \rightarrow \infty$. Only if D > 0 the eigenvalues λ_2 and λ_3 are complex eigenvalues, and within this condition as stated by relations in (6a) and (6b) we have two possibilities: h < 0 or h > 0. The following analytical considerations are encountered:

1) h<0. Following equations (6a) and (6b), this condition implies that m < 0 and n > 0. It could be easily demonstrated that, in this case, m+n has to be always positive. A direct consequence of this statement for equations (6a) and (6b) is that for h < 0, the eigenvalue λ_1 will dominate for $t \rightarrow \infty$ implying that for large time the flow is controlled by the real parts of the solution. Even if the expected 3D geometry may be very complicated in terms of the relative orientation of the vorticity vector with respect to the stretching directions, from the above consideration the following sub-cases could be described:

a) $TrL_{ij}>0$ (dilatant flow): in this case λ_1 bear always positive value. λ_2 and λ_3 could have positive or negative values depending if TrL_{ij} is respectively bigger or not than m+n;

b) $TrL_{ij}<0$ (collapsing flow): in this case we obtain the opposite possibility with respect to the previous one. In fact, λ_1 could have both positive and negative values, while λ_2 and λ_3 are always negative acting as implosing directions;

c) $TrL_{ij}=0$ (isochoric flow): in this case, λ_1 is always positive (being equivalent to m+n) but λ_1 and λ_2 are always negative. In the particular case of det (Lij) > 0 we have that $\lambda_1 \lambda_2 \lambda_3 > 0$ being (from eq. 6b) $\lambda_2 \lambda_3 = |\lambda_2|^2 |\lambda_3|^2 > 0$. We conclude that the real eigenvalue, λ_1 must always be positive.

2) h>0. From equations (6a), (6b) and (6c) we see that m < 0 and n > 0. The boundary condition is still $\Delta > 0$ but again, using the same logical approach as before, if we assume that h > 0, we obtain that m+n < 0. The difference with respect to the first case, where h < 0, is that in the case of m+n < 0 the two other real parts of the solution λ_2 and λ_3 are bigger than l_1 and hence will dominate at $t \rightarrow \infty$. This is equivalent to say that for large t, the asymptotic behaviour of the time dependent solution is now controlled by the couple of complex conjugate eigenvalues, so that now the ghostvectors define the flow pattern. This field of existence corresponds to the pulsating strain fields described by McKenzie (1979) and Weijermars (1993, 1997) in 2D, being only weakly perturbated by the eigenvector related to the real eigenvalue.

3) h=0. From equations (6a) and (6b) it is shown that m+n is also zero and, as a consequence, the solution of the general characteristic equation becomes strongly simplified and defined by three eigenvalues having the same real part. In particular, if $TrL_{ij} = 0$, the eigenvalues are totally complex and pattern is closed such as an ellipse or a circle.

Analysis and results

A detailed computation of all possible 3D flow patterns or displacement paths for such a deformation matrix is noisy and beyond the scope of this contribution. However, a criterion to obtain a first approximate visualization of the possible flow patterns in such a system is discussed. In order to achieve this, we find the eigenvalues ($\lambda_1 \lambda_2 \lambda_3$) and eigenvectors ν_i of the strain rate matrix Lij and then, using the general equation (3) describing the displacement path, try to understand and describe how particles move within flow controlled by complex eigenvalues. We describe some low-symmetry flow pattern, like the monoclinic or triclinic one, and underline the effect of strain on the flow development. Let's define a situation where the real eigenvalue λ_1 exceeds the real part of λ_2 and λ_3. This implies that h < 0 while it is necessary that TrLij ≠ 0. After large strain accumulations ($t \rightarrow \infty$) we expect that the real eigenvector related to l1 will control the deformation path as a repelling direction. Since this eigenvalue is real, a stable non-pulsating flow path will develop for $t \rightarrow \infty$. The two ghostvectors will constrain the geometry of the pattern but will not control the final flow accumulation. The pattern will develop from a closed or spiral pattern towards a helicoidal or "corkscrew" pattern. In the plane defined by the other two eigenvectors, a pulsating strain pattern, defined by the complex eigenvalues, is developed. In this condition at least two possible cases are envisaged; a) if the real part is zero, a closed pattern will develop (in this case the eigenvalues are also zero), and b) if the real part is not zero, a pulsating spiral pattern will develop.

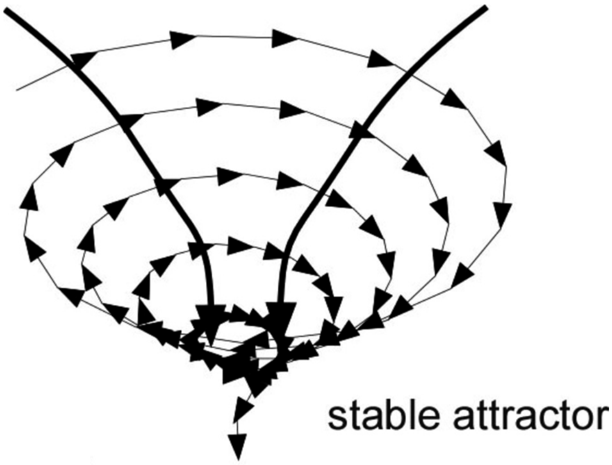

Figure 1. Flow pattern with real principal dominating eigenvalue showing a stable attracting eigenflow within an attracting spiral.

For case b),In this case, the following possibilities can be defined from equations (6a) and (6b):

i) For m+n > 2(Tr L_{ij})/3 we have $\lambda_1 > 0$ and the real parts of λ_2, λ_3 are < 0. In this situation, we should have a repelling eigenflow within a stable spiral or corkscrew pattern (Fig. 1).

ii) For m+n < 2(Tr L_{ij})/3, $\lambda_1 > 0$, the real parts of λ_2, λ_3 are both > 0. In this situation, the flow path have a third extruding eigenflow component within an unstable spiral or a corkscrew pattern (Fig. 2). If we assume in a second case that h > 0, being the real eigenvalue λ_1 smaller than λ_2 and λ_3, the following possibilities could be encountered:

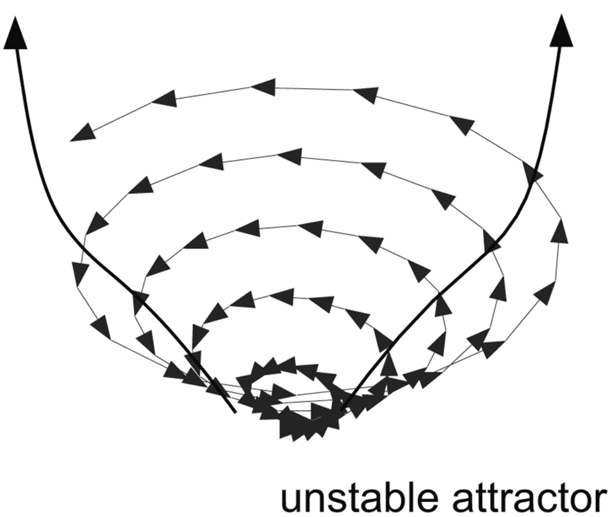

Figure 2. Flow pattern with real principal dominating eigenvalue showing an unstable eigenflow within a repelling spiral.

If m+n < 0 (being h < 0), $\lambda_1 < 0$, the real parts of λ_2 and λ_3 are both < 0. In this situation, we should have an attracting component overcoming the stable spiral path (Fig. 1), whilst if m+n < 0 (being h < 0), $\lambda_1 < 0$ but the real parts of λ_2 and λ_3 are > 0. In this situation, an attracting component should overcome the unstable spiral (Fig. 2). In this case, the geometry of the corkscrew pattern depends strongly on the absolute values of the real part that control the ghostvectors. However, the dominant pattern for $t \rightarrow \infty$ is the one controlled by the ghostvectors related to the complex conjugates λ_2 and λ_3. As a consequence, the flow patterns are dominated by the ghostvectors that define a pulsating pattern.

Triclinic flow

Another context where pulsating and transient strain could be registered by rock are the triclinic flows. Triclinic flows (Jiang and Williams, 1998) are the most natural examples of general 3D flow systems as the vorticity vector is not parallel to any ISA axes. Analytically, from the general equation (6a) describing the particle path, it is clear that a 3D geometry of a complex pattern is strongly controlled by the eigenvector or, more precisely, by the two ghost eigenvectors in combination with the real eigenvector. In the triclinic case, if the analysis is referred to an ISA system, the three eigenvectors loose their internal symmetry (monoclinic or orthorhombic) being function of the orientation angle β of the vorticity vectors (measured to the ISA). This implies that the new pattern shows a general 3D geometry (Fig. 3) deviating from the one showed in figure 1 showing a more complex and asymmetric 3D flow pattern. In figure 3, simple triclinic pattern is exposed. Moreover, as showed in table 1 and in detail in Iacopini *et al.* (2007), in the case of a triclinic flow, the field of existence of the complex eigenflow are far more important and larger with respect to the monoclinic one, because they do not develop only for $W_n > 1$ or within extruding flow (as in monoclinic case) but also for $W_n < 1$ within isochoric flows. This implies that the condition of $W_n > 1$, indicated by Ramberg (1974) and McKenzie (1979) as necessary condition to develop complex eigenvalues is valid only for monoclinic and general 2D flow but not for triclinic flow. Again, as showed in table 1 the critical vorticity number necessary to develop complex vectors can expected both at a dilatant and volume constant triclinic flow system. In the specific examples showed in table 1, the real eigenvalue is the biggest eigenvalue, implying a stable flow controlled by the principal eigenvalue.

Discussion

The theoretical approach presented in the previous paragraphs clearly point out that in a 3D flow system the existence of a stable or pulsating pattern does not simply depends on vorticity numbers. In fact, in case of triclinic flow, the condition of $W_n > 1$, crucial for general 2D and 3D monoclinic flow does not represent a threshold values on producing a flow with complex eigenvalues. Moreover the flow behaviour depends also on: a) the relative dominance of the real eigenvector with respect to the complex one, and b) the strain rate and total amount of strain accumulation ($t \rightarrow \infty$).

As indicated by the deformation path equations (3), the evolution of strain ellipse is quite complex as at least two of the strain axes rotate and pulsate. Depending on the type of flow (extruding or not), the third eigenvector could complicate the deformation path. If the real eigenvalue is larger than the real part of the other two complex eigenvalues, the eigenvector flow associated with such eigenvalue is expected to grow faster than the other two. Then, the initial flow pattern shows a pulsating but transient pattern, but after large strain accumulation, develops a stable pattern that is geometrically controlled by the attracting or repelling component defined by the real eigenvector (Fig. 4). If the real eigenvalue is smaller than the values of the other two complex eigenvectors, the flow pattern is similar to the 2D case with complex eigenvectors (Ramberg, 1974), since the real eigenvector will change the pulsating flow pattern only weakly. These behaviours seem to suggest that the concept of fabric stability is also strain and time dependent:

a) if the strain accumulation is low, theoretically it needs too much time to reach a stable repelling or attracting pattern and the rocks will continue to register a non stable pulsating pattern. This condition is expected in high grade domains where rocks do not localize deformation or strain rate accumulation is not very high. In this case, the extruding component does not overcome the pulsating fabric;

b) if strain accumulation is fast with a real eigenvector dominating and there is enough strain accumulation,

An = 0, Tn = 0		An = 0, Tn = 0.5		An = 0, Tn = 0.7		An = 0, Tn = 1	
α	Wd	α	Wd	α	Wd	α	Wd
0.1	0.94	0.1	0.99	0.1	0.99	0.1	0.99
0.2	0.88	0.2	0.98	0.2	0.99	0.2	0.99
0.3	0.82	0.3	0.98	0.3	0.99	0.3	0.99
0.4	0.78	0.4	0.96	0.4	0.99	0.4	0.99

An = 0.5, Tn = 0		An = 0.5, Tn = 0.5		An = 0.5, Tn = 0.7		An = 0.5, Tn = 1	
α	Wd	α	Wd	α	Wd	α	Wd
0.1	0.75	0.1	0.88	0.1	0.91	0.1	0.99
0.2	0.64	0.2	0.81	0.2	0.81	0.2	0.90
0.3	0.55	0.3	0.73	0.3	0.79	0.3	0.86
0.4	0.48	0.4	0.69	0.4	0.75	0.4	0.82

An = 0.7, Tn = 0		An = 0.7, Tn = 0.5		An = 0.7, Tn = 0.7		An = 0.7, Tn = 1	
α	Wd	α	Wd	α	Wd	α	Wd
0.1	0.57	0.1	0.69	0.1	0.76	0.1	0.79
0.2	0.45	0.2	0.61	0.2	0.64	0.2	0.70
0.3	0.37	0.3	0.53	0.3	0.56	0.3	0.62
0.4	0.32	0.4	0.48	0.4	0.51	0.4	0.55

An = 0.9, Tn = 0		An = 0.9, Tn = 0.5		An = 0.9, Tn = 0.7		An = 0.9, Tn = 1	
α	Wd	α	Wd	α	Wd	α	Wd
0.1	0.29	0.1	0.36	0.1	0.39	0.1	0.44
0.2	0.19	0.2	0.27	0.2	0.30	0.2	0.33
0.3	0.15	0.3	0.20	0.3	0.23	0.3	0.26
0.4	0.14	0.4	0.17	0.4	0.22	0.4	0.22

Table 1. Threshold vorticity number limits within different triclinic flow types. : orientation of the vorticity vector with respect to ISA; An: dilatancy parameters; Tn: extruding parameters; Wd: sectional vorticity numbers.

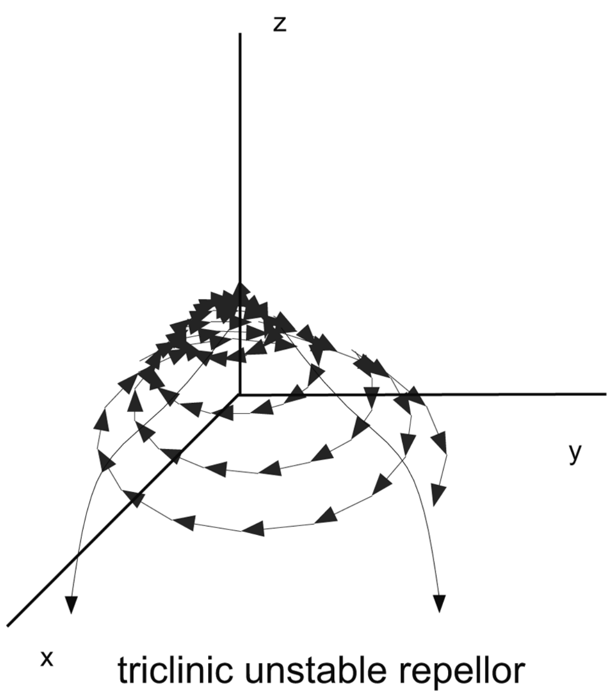

Figure 3. A possible triclinic example in ISA reference system: triclinic stable attractor. Spiral pattern with an oblique disposition of the real attracting eigenvector.

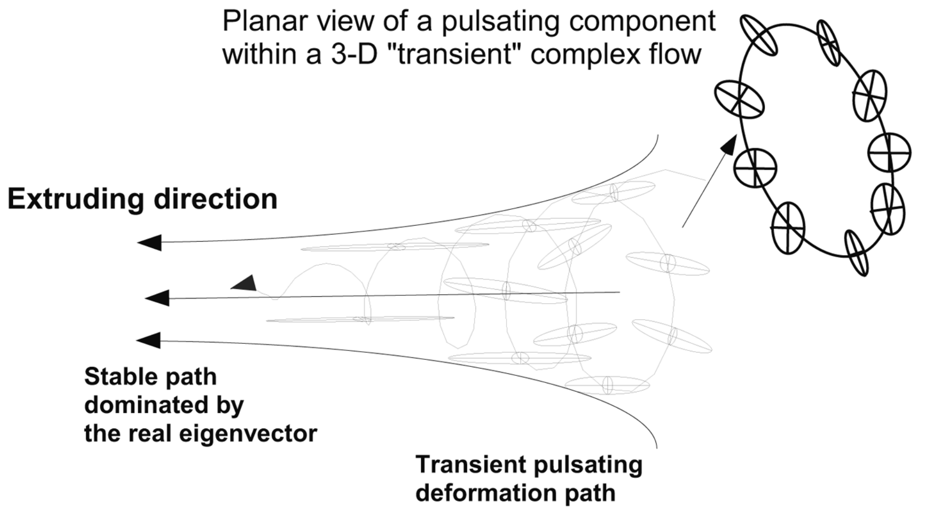

Figure 4. Sketch of a possible 3D extruding flow path (with h<0 and Δ>0). The 3D ellipsoid indicates the progressive ellipsoid pulsating history from the transient initial pulsating pattern to the more stable extruding pattern. Note the stronger stretching component induced by the major real eigenvector overcoming the pulsating pattern during the last accumulation event.

the flow pattern could start with a transient pulsating strain and then rapidly merge to a stable final flow pattern (Fig. 4). In extruding contexts, the initial strengthening-weakening fabric development could prelude to a more simple high strain stable condition.

Conclusion

Within a 3D deformed homogeneous and steady state flow system, the complex eigenvalues produce ghostvectors that could not attract or stabilize any flow system as the correspondent real one. In order to better define the 3D flow pattern characterized by ghostvectors, we investigated the possible eigenvalues distribution, discussed the nature of some related flow pattern and predicted some new possible stable flow pattern with vorticity number $W_n > 1$. The results of the present work convey two messages:

a) in a 3D flow system, unlike the 2D system, a field with all complex eigenvalues cannot exist because there is always almost one real eigenvalue controlling the flow system. Moreover, in case of triclinic flow, the condition of $W_n > 1$ does not represents a necessary condition to develop complex eigenvalues;

b) according to the kinematic calculations we suggest that after a certain amount of strain accumulation if the real eigenvalues is bigger than the real part of the two other complex conjugate eigenvalues, a non-pulsating and stable fabric could be expected.

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